

# $k$ -disjunctive cuts and a finite cutting plane algorithm for general mixed integer linear programs

Markus Jörg

Technische Universität München, Zentrum Mathematik  
Boltzmannstraße 3, 85747 Garching bei München, Germany  
joerg@ma.tum.de

July 26, 2007

**Abstract:** In this paper we give a generalization of the well known split cuts of Cook, Kannan and Schrijver [5] to cuts which are based on multi-term disjunctions. They will be called  $k$ -disjunctive cuts. The starting point is the question what kind of cuts is needed for a finite cutting plane algorithm for general mixed integer programs. We will deal with this question in detail and derive cutting planes based on  $k$ -disjunctions related to a given cut vector. Finally we will show how a finite cutting plane algorithm can be established using these cuts in combination with Gomory mixed integer cuts.

## 1 Introduction

In this paper we will deal with cutting planes and related algorithms for general mixed integer linear programs (MILP). As most of the results will be derived by geometric arguments we focus on programs that are given by inequality constraints, i.e.

$$\begin{aligned} \max \quad & cx + hy \\ & Ax + Gy \leq b \\ & x \in \mathbb{Z}^p \end{aligned} \tag{1}$$

where the input data are the matrices  $A \in \mathbb{Q}^{m \times p}$ ,  $G \in \mathbb{Q}^{m \times q}$ , the column vector  $b \in \mathbb{Q}^m$  and the row vectors  $c \in \mathbb{Q}^p$ ,  $h \in \mathbb{Q}^q$ . Moreover we denote by the polyhedra  $P = \{(x, y) : Ax + Gy \leq b\} \subset \mathbb{R}^{p+q}$  and  $P_I = \text{conv}(\{(x, y) \in P : x \in \mathbb{Z}^p\}) \subset \mathbb{R}^{p+q}$  the feasible domains of the LP relaxation and the (mixed) integer hull of a given MILP, respectively. We call

a MILP bounded, if the polyhedron  $P$  is bounded. We will also need the projection  $\text{proj}_X(P) := \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q : (x, y) \in P\}$  of the polyhedron  $P$  on the space of the integer variables.

By a cutting plane for  $P$  we understand an inequality  $\alpha x + \beta y \leq \gamma$  with row vectors  $\alpha \in \mathbb{Q}^p, \beta \in \mathbb{Q}^q$  which is valid for  $P_I$  but not for  $P$ . Using cutting planes gives a simple idea of how to solve a general MILP: Solve the LP relaxation of the MILP. If the optimal solution is feasible, i.e. satisfies the integrality constraint, an optimal solution is found. Otherwise find a valid cutting plane that cuts off the current solution and repeat. But unlike the pure integer case no finite exact cutting plane algorithm is known for general MILP. Therefor we remark that most cutting planes for general MILP such as e.g. Gomory mixed integer cuts [7] or mixed integer rounding cuts [11] are special cases of or equivalent to split cuts [5]. This fact and more detailed relations between these and other cuts are stated in [6]. Here a split cut is defined as a cutting plane  $\alpha x + \beta y \leq \gamma$  for  $P$  with the additional property that there exists  $d \in \mathbb{Z}^p, \delta \in \mathbb{Z}$  such that  $\alpha x + \beta y \leq \gamma$  is valid for all  $(x, y) \in P$  which satisfy the split disjunction  $dx \leq \delta$  or  $dx \geq \delta + 1$ . So split cuts are defined not constructively but by a property, only. Now one can see in the following 'classical' example of Cook, Kannan and Schrijver [5] that split cuts are not sufficient for solving a general MILP in finite time.

**Example 1.** *The MILP*

$$\begin{aligned} \max y \\ -x_1 + y &\leq 0 \\ -x_2 + y &\leq 0 \\ x_1 + x_2 + y &\leq 2 \\ x_1, x_2 &\in \mathbb{Z} \end{aligned}$$

*has the optimal objective function value 0 but the problem cannot be solved by any algorithm that uses split cuts, only. A proof of this statement in a more general context is given in Lemma 3.*

On the other hand, as positive results in the context of cutting plane algorithms for MILP we can only give the following two special cases: For mixed 0-1 programs split cuts are sufficient for generating the integer hull  $P_I$  of a given polyhedron  $P$ . See e.g. [11] in the context of mixed integer rounding cuts or [3] in the more recent representation of lift-and-project cuts. For general MILP, there only exists a finite approximation algorithm of Owen and Mehrotra [12] which finds a feasible  $\epsilon$ -optimal solution and uses simple split cuts, that means split cuts to disjunctions  $x_i \leq \delta \vee x_i \geq \delta + 1$ .

So as split cuts fail in the design of a finite cutting plane algorithm for general MILP we want to generalize this approach to cuts that are based on multi-term disjunctions. Therefor we start in section 2 with the introduction of  $k$ -disjunctive cuts and some of its basic properties. Afterwards we look at the approximation properties of the  $k$ -disjunctive closures and deal with the question what kind of cuts is needed for an exact finite cutting

plane algorithm both in general and in special cases. Finally we derive a  $k$ -disjunctive cut according to a given cut vector. In section 3 we turn to algorithmic aspects and give a way of how a finite cutting plane algorithm for general MILP can be designed using  $k$ -disjunctive cuts in connection with the well known mixed integer Gomory cuts. Finally we will discuss the algorithm and give some interpretations.

## 1.1 Preliminaries

Here we repeat two basic results that we will need during this paper. The first one deals with the computation of the projection  $\text{proj}_X(P)$ , the second one with the convergence of the mixed integer Gomory algorithm in a special case.

**Lemma 1.** *Let a polyhedron  $P = \{(x, y) : Ax + Gy \leq b\}$  be given. Then*

$$\text{proj}_X(P) = \{x \in \mathbb{R}^p : v^r Ax \leq v^r b, \forall r \in R\},$$

where  $\{v^r\}_{r \in R}$  is the set of extreme rays of the cone  $Q := \{v \in \mathbb{R}^m : G^T v = 0, v \geq 0\}$ .

*Proof.* The statement follows by applying the Farkas Lemma, see e.g. [10], I.4.4.  $\square$

Next we look at the usual mixed integer Gomory algorithm [7]. Although the algorithm does in general not even converge to the optimum, the special case in which the optimal objective function value can be assumed to be integer, e.g. the case of  $h = 0$ , can be solved finitely using the algorithm. In detail we have the following

**Theorem 1.** *Let a bounded MILP (1) be given. Then the mixed integer Gomory algorithm terminates finitely with an optimal solution or detects infeasibility under the following conditions:*

1. *One uses the lexicographic version of the simplex algorithm for solving the LP relaxation.*
2. *The optimal objective function value is integral.*
3. *A least index rule is used for cut generation, i.e. the mixed integer Gomory cut according to the first variable  $x_j$ , that is fractional in the current LP solution, is added to the program. Here  $x_0$  corresponds to the objective function value.*

Using the last theorem, it is obvious that we can check in finite time if there is a feasible point in a polytope with a given (rational) objective function value, as by scaling it can be always assumed that the optimal objective function value is integral. This is expressed in the following

**Corollary 1.** *Let a bounded MILP (1) with the additional constraint  $cx + hy = \gamma$  be given. Then the mixed integer Gomory algorithm terminates finitely with a feasible solution or detects infeasibility.*

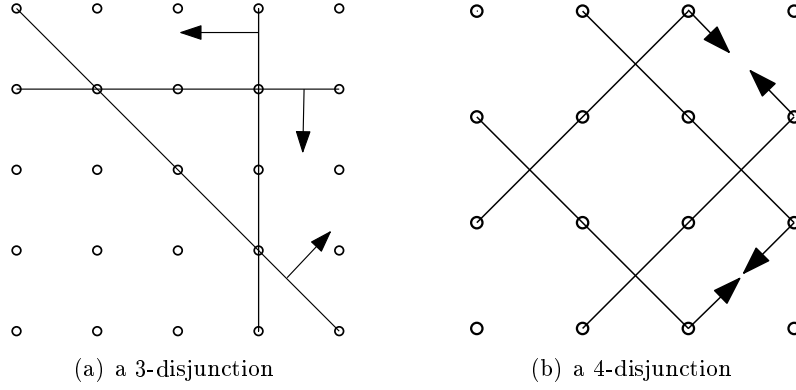


Figure 1: Examples for  $k$ -disjunctions in  $\mathbb{R}^2$

## 2 $k$ -disjunctive cuts

### 2.1 Basic definitions and properties

In analogy to the definition of a split cut based on a split disjunction we now define a  $k$ -disjunctive cut that is based on a  $k$ -disjunction that contains every integral vector.

**Definition 1.** Let  $k \geq 2$  be a natural number,  $d^1, \dots, d^k \in \mathbb{Z}^p$  integral vectors and  $\delta^1, \dots, \delta^k \in \mathbb{Z}$ . Then we call the inequalities  $d^1x \leq \delta^1, \dots, d^kx \leq \delta^k$  a  $k$ -disjunction, if for all  $x \in \mathbb{Z}^p$  there is an  $i \in \{1, \dots, k\}$  with  $d^ix \leq \delta^i$ . In this case we write  $D(k, d, \delta)$  with  $d = (d^1, \dots, d^k), \delta = (\delta^1, \dots, \delta^k)$  for the  $k$ -disjunction.

We note that we do not require the vectors  $d^i, \delta^i$  to be different. So every  $l$ -disjunction is also a  $k$ -disjunction for  $l < k$ . Especially every split disjunction is also a  $k$ -disjunction. Moreover every  $k$ -disjunction is a cover of  $\mathbb{Z}^p$  by definition.

**Definition 2.** Let  $P \subset \mathbb{R}^{p+q}$  be a polyhedron and  $\alpha x + \beta y \leq \gamma$  be a cutting plane. Then  $\alpha x + \beta y \leq \gamma$  is called a  $k$ -disjunctive cut for  $P$ , if there exists a  $k$ -disjunction  $D(k, d, \delta)$  with

$$(x, y) \in P : \alpha x + \beta y > \gamma \implies d^i x > \delta^i, \forall i \in \{1, \dots, k\}.$$

Of course every  $k$ -disjunctive cut for  $P$  is valid for  $P_I$  by definition. According to the remark after Definition 1 every  $l$ -disjunctive cut is also a  $k$ -disjunctive cut for  $l < k$ . So every split cut is a  $k$ -disjunctive cut.

**Definition 3.** Let  $P \subset \mathbb{R}^{p+q}$  be a polyhedron. Then the intersection of all  $k$ -disjunctive inequalities is called the  $k$ -disjunctive closure of  $P$  and denoted by  $P_k^{(1)}$ . Analog the  $i$ -th  $k$ -disjunctive closure  $P_k^{(i)}$  of  $P$  is defined as the  $k$ -disjunctive closure of  $P_k^{(i-1)}$ . In the special case of  $k = 2$  we will also write  $P^{(i)}$  instead of  $P_2^{(i)}$ .

We want to remark that it is not evident if the  $k$ -disjunctive closure  $P_k^{(1)}$  for a given polyhedron  $P$  is again a polyhedron in the case of  $k \geq 3$ . The both proofs of this property for the split closure [1] and [5] cannot be applied to the more general case. However, we will not further deal with this question, as our results in the following are independent of this property. We further remark that Definition 2 also applies in a natural way to closed convex sets  $P$ . Therewith it is guaranteed that the definition of the  $i$ -th  $k$ -disjunctive closure  $P_k^{(i)}$  of a polyhedron  $P$  is well defined.

A valid cut to a given  $k$ -disjunction can be computed as intersection cut to any basis solution of the LP-relaxation that is not contained in the disjunction according to [2]. In the case of  $k = 2$ , Andersen, Cornuéjols and Li have shown [1] that intersection cuts are sufficient to describe all cuts to a given split disjunction. This result is not true for general  $k$ -disjunctions. Here not every valid  $k$ -disjunctive cut to a given disjunction is equal to or dominated by an intersection cut. This can be seen in the following

**Example 2.** We look at the polyhedral cone  $C \in \mathbb{R}^{2+1}$  with apex  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  that is defined by

$$\begin{aligned} -x_1 + y &\leq 0 \\ -x_2 + y &\leq 0 \\ x_1 + y &\leq 1 \\ x_2 + y &\leq 1. \end{aligned}$$

Then  $y \leq 0$  is a 4-disjunctive cut for  $C$  to the 4-disjunction  $D := \{x_1 + x_2 \geq 2, x_1 - x_2 \geq 1, -x_1 + x_2 \geq 1, -x_1 - x_2 \geq 0\}$ . The set of all bases is given by any three of the above constraints. Computing the four relating intersection cuts to the 4-disjunction  $D$  we get that the point  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{6})$  is valid for the four cuts and so  $y \geq \frac{1}{6}$  has to be satisfied.

Although the properties of general  $k$ -disjunctive cuts are more involved than in the case of split cuts, an investigation of these cuts is useful because every valid cutting plane for a given polyhedron is a  $k$ -disjunctive cut for some  $k$ .

**Lemma 2.** Let  $P \subset \mathbb{R}^{p+q}$  be a polyhedron and  $\alpha x + \beta y \leq \gamma$  be a valid cutting plane. Then  $\alpha x + \beta y \leq \gamma$  is a  $k$ -disjunctive cut for some  $k \in \mathbb{N}$ .

*Proof.* Let  $P$  be a polyhedron and  $\alpha x + \beta y \leq \gamma$  be a valid cutting plane. The set  $M$  that is cut off by the above cutting plane is given by

$$M := \{(x, y) \in P : \alpha x + \beta y > \gamma\}.$$

$M$  contains no feasible points of  $P_I$ . So we have  $x \notin \mathbb{Z}^p$  for  $(x, y) \in M$ . By Lemma 1 the projection of  $M$  can be expressed as

$$\text{proj}_X(M) = \{x \in \mathbb{R}^p : A^e x \leq b^e, A^l x < b^l\}$$

with wlog integral matrices  $A^e, A^l$  with rows  $a_i$  satisfying  $\gcd(a_i) = 1$  and integer vectors  $b^e, b^l$ . We modify the coefficients of the vectors  $b^e, b^l$  by

$$\begin{aligned}\tilde{b}_i^e &= \lfloor b_i^e \rfloor + 1, \\ \tilde{b}_i^l &= \lceil b_i^l \rceil.\end{aligned}$$

Altogether we get that  $\alpha x + \beta y \leq \gamma$  is a  $k$ -disjunctive cut according to  $D(k, -(A^e, A^l), -(\tilde{b}^e, \tilde{b}^l))$ . □

It is our goal to compute the mixed integer hull of a given polyhedron using  $k$ -disjunctive cuts. Of course this should be done 'as simple as possible', what means that both the number  $k$  of hyperplanes needed for the disjunctions and the number of iterations in a cutting plane procedure should be small. At least the latter property can be easily realized as the next theorem shows.

**Theorem 2.** *Let  $P \subset \mathbb{R}^{p+q}$  be a polyhedron. Then  $P_I = P_{2^p}^{(1)}$ .*

*Proof.* We will show that every valid inequality  $\alpha x + \beta y \leq \gamma$  for  $P_I$  is a  $2^p$ -disjunctive cut for  $P$ . This is sufficient for the theorem. Using Lemma 2 we get that  $\alpha x + \beta y \leq \gamma$  is a  $k$ -disjunctive cut with a related disjunction  $D(k, d, \delta)$ . So the claim is shown for  $k \leq 2^p$ . Otherwise the number of inequalities of the disjunction can be reduced until the required limit of  $2^p$ . Since  $D(k, d, \delta)$  is a  $k$ -disjunction we have

$$\forall x \in \mathbb{Z}^p \exists i \in \{1, \dots, k\} : d^i x \leq \delta^i.$$

On the other hand we take the set of all integral vectors with the property  $d^i x = \delta^i$  for a given  $i \in \{1, \dots, k\}$ . Either it exists now a vector  $\bar{x} \in \mathbb{Z}^p$  with  $d^i \bar{x} = \delta^i$  and  $d^j \bar{x} > \delta^j$ ,  $\forall j \in \{1, \dots, k\} \setminus \{i\}$ , or we can expand the disjunction by setting the right hand side of the inequality to  $\delta^i - 1$  and repeat this consideration. This may also lead to the case that the inequality can be dropped. Therewith we can restrict ourselves to disjunctions with the additional condition:

$$\forall i \in \{1, \dots, k\} \exists x^i \in \mathbb{Z}^p : d^i x^i = \delta^i \wedge d^j x^i > \delta^j, \forall j \in \{1, \dots, k\} \setminus \{i\}$$

The set  $\text{conv}(\{x^1, \dots, x^k\})$  contains except for its vertices  $\{x^1, \dots, x^k\}$  no more integral vectors: Assumed that there was another integral vector  $z \in \text{conv}(\{x^1, \dots, x^k\})$  we had  $d^i z \leq \delta^i$  for an  $i$ . This contradicts the definition of the vectors  $x^i$ . So we have constructed a set that contains exactly  $k$  integer points as its vertices. This will lead to a contradiction for  $k > 2^p$ . Then we had at least two vertices  $v, w$  with the additional property that each component  $v_i, w_i$ ,  $i \in \{1, \dots, n\}$  of both vectors is either even or odd. So  $\frac{1}{2}(v + w)$  is an integral vector which is contained in  $\text{conv}(\{x^1, \dots, x^k\})$ . This is a contradiction to the properties of the set. □

We look at an easy example to see that  $2^p$ -disjunctive cuts are needed in general to compute the mixed integer hull of a polyhedron in one step.

**Example 3.** We take the  $p$ -dimensional unit cube  $C = [0; 1]^p$  and define the polyhedron  $Q$  by

$$Q = \{x : ax \leq \max_{x \in C} ax, a \in \{-1, 1\}^p\}.$$

Next we embed  $Q$  in the  $\mathbb{R}^{p+1}$  and define the polyhedron

$$P = \text{conv} \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix}, \frac{1}{2}\mathbf{1} \right\}, x \in Q.$$

Of course it is  $P_I = C$  and the only valid  $k$ -disjunction for the cutting plane  $x_{p+1} \leq 0$  is defined by the facets of  $Q$  itself.

As Theorem 2 shows, the mixed integer hull of a general polyhedron can be 'easily' generated with  $2^p$ -disjunctive cuts in theory. Of course for practical issues the use of disjunctions with an exponential number of defining hyperplane is very expensive. So we will deal in the following with the second question we mentioned above, i.e. what kind of  $k$ -disjunctive cuts we need at least in computing the mixed integer hull using a repeated application of  $k$ -disjunctive cuts.

## 2.2 Approximation property of split cuts

Before we further analyze which cuts we need to solve a MILP exactly, we will deal with the approximation properties of  $k$ -disjunctive cuts. We repeat that already using split cuts is sufficient to approximate the optimal objective function value of any MILP arbitrarily exact. Therefor look at the series  $(\gamma^{(i)})_{i \in \mathbb{N}}$  of objective function values that is given by

$$\gamma^{(i)} := \max\{cx + hy \mid (x, y) \in P^{(i)}\} \quad (2)$$

for an arbitrary objective function  $cx + hy$  that is bounded over the polyhedron  $P$ . In detail we get the following

**Theorem 3.** Let  $P \subset \mathbb{R}^{p+q}$  be a polytope,  $cx + hy$  an objective function,  $\gamma^* = \max\{cx + hy \mid (x, y) \in P_I\}$  and  $\gamma^{(i)}$  as defined in (2). Then for all  $\epsilon > 0$  there is an  $i_0 \in \mathbb{N}$  with  $|\gamma^{(i_0)} - \gamma^*| < \epsilon$ .

*Proof.* A proof of this statement in a slightly different form using a repeated variable disjunction can be found in the paper [12] of Owen and Mehrotra. Moreover the algorithm in this paper also gives a constructive proof.  $\square$

As we can approximate any optimal objective function value arbitrarily exact using split cuts, the use of general  $k$ -disjunctive cuts becomes necessary for determining exact solutions, only. Moreover we want to remark that in practical applications already optimizing over the first split closure often gives a good approximation of the optimal objective function value. This was in detail investigated by Balas and Saxena [4] for instances from the MIPLIB 3.0 and several other classes of structured MILP.

### 2.3 Solving MILP exactly

We now get back to the question what kind of cuts is needed to solve a general MILP exactly. As we will see, this depends on the structure of the projection of the solution space on the  $x$ -space of integral variables. For example the important special case of the solution space being a vertex can be solved just using split cuts. However, we will see that in general the required number of disjunctive hyperplanes is exponential in the dimension of the integer space. We start with the case that the solution set contains relative interior integer points.

**Theorem 4.** *Let  $P \subset \mathbb{R}^{p+q}$  be a polyhedron,  $cx+hy$  an over  $P$  bounded objective function and  $\gamma^* = \max\{cx + hy \mid (x, y) \in P_I\}$ . If*

$$\text{relint}(\text{proj}_X(\{(x, y) \in P_I : cx + hy = \gamma^*\})) \cap \mathbb{Z}^p \neq \emptyset$$

*then there is a  $k \in \mathbb{N}$  with  $\max\{cx + hy \mid (x, y) \in P^{(k)}\} = \gamma^*$ .*

*Proof.* If  $\max\{cx + hy \mid (x, y) \in P\} = \gamma^*$  there is nothing to show, so let  $\max\{cx + hy \mid (x, y) \in P\} > \gamma^*$ . That means especially that  $\text{int}(\text{proj}_X(M)) \cap \mathbb{Z}^p = \emptyset$  where  $M := P_I \cap \{(x, y) : cx + hy = \gamma^*\}$  denotes the solution set. Moreover let  $x^* \in \text{relint}(\text{proj}_X(M)) \cap \mathbb{Z}^p$ . To proof the claim we have to show that  $cx + hy \leq \gamma^*$  is a split cut for one of the polyhedra  $P^{(k)}$ ,  $k \in \mathbb{N}$ .

Let  $A_I x + G_I y \leq b_I$  denote these inequalities in the representation of the mixed integer hull that constrain the set  $M$ . With Theorem 3 we get

$$\lim_{k \rightarrow \infty} \max\{a_{I,i}x + g_{I,i}y \mid (x, y) \in P^{(k)}\} = b_{I,i}. \quad (3)$$

Moreover  $x^*$  lies in the boundary of the projection  $\text{proj}_X(M^{(k)})$  of each set  $M^{(k)} := P^{(k)} \cap \{(x, y) : cx + hy \geq \gamma^*\}$ . As  $M^{(k+1)} \subseteq M^{(k)}$  there exists an inequality  $px \leq \pi$  that is valid for all of the sets  $\text{proj}_X(M^{(k)})$  with the additional property

$$px = \pi, \forall x \in \text{proj}_X(M), \quad (4)$$

as  $x^* \in \text{relint}(\text{proj}_X(M))$ . If we combine (3) and (4) we get as direct consequence that  $cx + hy \leq \gamma^*$  is a split cut to the disjunction  $D(p, \pi)$  for some  $P^{(n)}$ ,  $n \in \mathbb{N}$ .  $\square$

After we have seen that split cuts are even sufficient for solving an important class of MILP exactly, we turn to the general situation. The idea for finite convergence using  $k$ -disjunctive cuts in general consists of the basic principle that there has to exist a  $k$ -disjunction  $D$  so that the interior of the projection  $\text{proj}_X(M)$  of the solution set is not contained in  $D$ . If no appropriate  $k$ -disjunction exists for all closures  $P_k^{(i)}$  then we cannot achieve a finite algorithm using  $k$ -disjunctive cuts. On the other hand, if this condition is satisfied for each face of the solution set, finite convergence can be shown in the general case.



**Theorem 5.** Let  $P \subset \mathbb{R}^{p+q}$  be a polyhedron,  $cx + hy$  an over  $P$  bounded objective function,  $\gamma^* = \max\{cx + hy \mid (x, y) \in P_I\}$  and  $M := P_I \cap \{(x, y) : cx + hy = \gamma^*\}$ . If there exists for both  $M$  and all its faces  $f \in F$  with  $\text{relint}(\text{proj}_X(f)) \cap \mathbb{Z}^p = \emptyset$  a  $k$ -disjunction  $D_f(k, d, \delta)$  with the property

$$x \in \text{relint}(\text{proj}_X(f)) \implies x \notin D_f,$$

then there exists a  $n \in \mathbb{N}$  with  $\max\{cx + hy \mid (x, y) \in P_k^{(n)}\} = \gamma^*$ .

*Proof.* We prove the claim by induction over the dimension  $l$  of the solution set  $M$ . We start with  $l = 0$ . In this case  $k = 2$  can always be chosen and the result is a special case of Theorem 4. We assume now that the claim is true for  $l - 1, l \in \mathbb{N}$ .

So let  $M$  be the solution set of  $\max\{cx + hy \mid (x, y) \in P_I\}$  and  $\dim(M) = l$ . Moreover let for  $k \in \mathbb{N}$  exist a  $k$ -disjunction  $D(k, d, \delta)$  for  $M$  according to the assumption. We proof that  $cx + hy \leq \gamma^*$  is a  $k$ -disjunctive cut to the disjunction  $D$  for one of the sets  $P_k^{(n)}$ . Therefor we show that it exists a  $n \in \mathbb{N}$  such that  $(P_k^{(n)} \cap \{(x, y) : cx + hy > \gamma^*\}) \cap D = \emptyset$ . With the disjunction  $D$  we define the polyhedron  $Q := \{(x, y) : dx \geq \delta \wedge cx + hy = \gamma^*\}$ . As  $cx + hy = \gamma^*$  is a supporting hyperplane of  $P_I$ , there exists  $(\hat{x}, \hat{y})$  with  $c\hat{x} + h\hat{y} > \gamma^*$  and  $\hat{x} \notin D$  such that each of the inequalities  $c_fx + h_fy \leq \gamma_f$  defined by  $(\hat{x}, \hat{y})$  and a facet  $f$  of  $Q$  is valid for  $P_I$ . All inequalities  $c_fx + h_fy \leq \gamma_f$  at most support  $M$  in an under dimensional face. So it follows either by induction hypothesis or by Theorem 3 that the inequalities  $c_fx + h_fy \leq \gamma_f$  are valid for some  $P_k^{(n)}$ . Herewith, the condition  $(P_k^{(n)} \cap \{(x, y) : cx + hy > \gamma^*\}) \cap D = \emptyset$  is satisfied and the theorem is proven.  $\square$

We remark that it is necessary to involve all the faces of the solution set in the last theorem, as the following example shows.

**Example 4.** We define the polyhedron  $P \subset \mathbb{R}^{3+1}$  through the vertices

$$\begin{aligned} &(0, 0, 0, 0), (2, 0, 0, 0), (0, 2, 0, 0) \\ &(0, 0, 1, 0), (2, 0, 1, 0), (0, 2, 1, 0) \\ &\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

For the objective function vector  $(0, 0, 0, 1)$  we get that  $M$  is contained in a split disjunction, whereas the cut according to the face  $x_3 = 0$  is a 3-disjunctive cut, only.

We now deal with the question what cuts we need to solve a general MILP. Therefor we use special sets that can arise as solution sets of MILP to give a lower bound of the required number of disjunction terms. The idea is based on a generalization of Example 1.

**Lemma 3.** *Let  $P \subset \mathbb{R}^{p+q}$  be a polyhedron,  $cx + hy$  an over  $P$  bounded objective function,  $\gamma^* = \max\{cx + hy \mid (x, y) \in P_I\} > \max\{cx + hy \mid (x, y) \in P\}$  and  $M := P_I \cap \{(x, y) : cx + hy = \gamma^*\}$ . If  $\text{proj}_X(M) \subset \mathbb{R}^p$  has  $k$  facets with each containing a relative interior integer point, then  $\max\{cx + hy \mid (x, y) \in P_{k-1}^{(i)}\} > \gamma^*, \forall i \in \mathbb{N}$ .*

*Proof.* Let  $\text{proj}_X(M)$  be given with relative interior points  $\{x^1, \dots, x^k\} \in \mathbb{Z}^p$  that are contained in pairwise different facets. Then we have  $x^{ij} := \frac{1}{2}(x^i + x^j) \in \text{int}(\text{proj}_X(M))$  for all  $i, j, i \neq j$ . By presumption there exists  $(x^{ij}, y^{ij}) \in P$  with  $cx^{ij} + hy^{ij} > \gamma^*$ . Moreover at least one of the points  $(x^{ij}, y^{ij})$  is not cut off by an arbitrary  $k-1$ -disjunctive cut. So the cut is valid for the set  $Q^{ij} := \text{conv}((x^{ij}, y^{ij}), P_I)$ . As each cut can be classified by this property we get that  $\bigcap_{i \neq j} Q^{ij} \subseteq P_{k-1}^{(1)}$ . It is clear that  $\bigcap_{i \neq j} Q^{ij}$  contains a point  $(x, y)$  with  $cx + hy > \gamma^*$ . As  $P_{k-1}^{(1)}$  satisfies all presumptions and the solution set  $M$  does not change, the proof follows by induction.  $\square$

Therewith we can show now that we need cutting planes to an in the dimension  $p$  exponential number of disjunctive terms to solve a MILP in general.

**Theorem 6.** *Let  $P$  be a polyhedron,  $cx + hy$  be an over  $P$  bounded objective function with  $\max\{cx + hy : (x, y) \in P\} = \gamma^*$ . Then in general*

$$\max\{cx + hy : (x, y) \in P_{2^{p-1}+1}^{(n)}\} > \gamma^*, \forall n \in \mathbb{N}.$$

*Proof.* Using Lemma 3 it is sufficient to give an integer polytope  $Q \subset \mathbb{R}^p$  with  $p = n + 1$  and at least  $2^n + 2$  facets that contains no interior integer point but in each facet a relative interior integer point. Therefor we define  $Q$  as the set of all  $(x, x_{n+1}) \in \mathbb{R}^{n+1}$  with the property:

$$\begin{aligned} ax - \pi(a)x_{n+1} &\leq 1, \quad a \in \{\pm 1\}^n \\ 0 &\leq x_{n+1} \leq 2 \end{aligned}$$

with  $\pi(a) := |\{i \in \{1, \dots, n\} : a_i = 1\}| - 1$ . We show that  $Q$  has the desired properties. Its vertices are contained in the hyperplane  $x_{n+1} = 0$  or  $x_{n+1} = 2$ . For  $x_{n+1} = 0$  the related polytope is the  $n$ -dimensional cross polytope. For  $x_{n+1} = 2$  the related polytope is generated by the vertices  $\mathbf{1} + (n-1)u_i$ . The last property follows from the fact that  $ax \leq 1$  is active for a vector  $\pm u_i$  if, and only if  $ax \leq 1 + 2\pi(a)$  is active for  $\mathbf{1} \pm (n-1)u_i$  by definition of  $\pi(a)$ . So  $Q$  is integer.

Let  $(z, 1) \in \mathbb{Z}^{n+1}$  be given. We take the side constraint  $ax \leq 1 + \pi(a)$  with  $a_i = 1 \iff z_i > 0, i \in \{1, \dots, n\}$  and get

$$az = \sum_{1 \leq i \leq n} |z_i| \geq \sum_{z_i > 0} z_i \geq |\{i \in \{1, \dots, n\} : a_i = 1\}| = 1 + \pi(a).$$

So  $(z, 1)$  is no interior point of  $Q$ . Moreover we can see that  $(z, 1) \in Q$  for  $z \in \{0, 1\}^n$  and that  $(z, 1)$  is a relative interior point of the facet  $ax - \pi(a)x_{n+1} \leq 1$  for  $a_i = 1 \iff z_i = 1, i \in \{1, \dots, n\}$ . As  $\mathbf{0}$  and  $\mathbf{1}$  are relative interior point of  $x_{n+1} = 0$  and  $x_{n+1} = 2$ ,  $Q$  has all properties.  $\square$

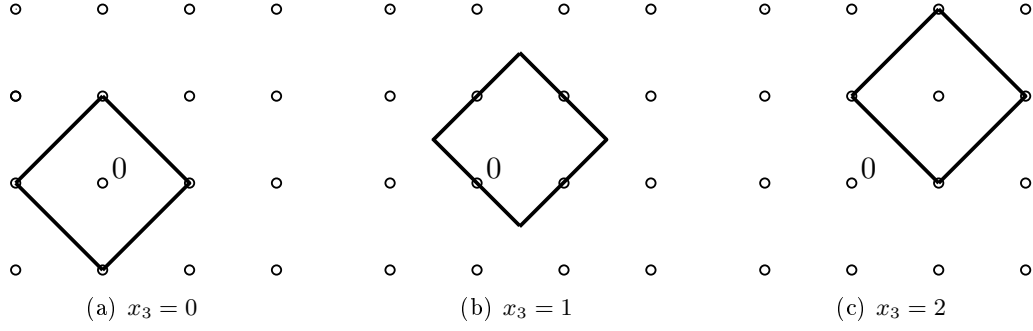


Figure 2: The set  $Q$  constructed in Theorem 6 for  $p = 3$ , projected on the  $x_3$ -space

So we have proven that in general at least  $2^{p-1} + 2$ -disjunctive cuts are required to solve a MILP exactly in a finite number of steps. We remark that we have an upper bound of  $2^p$  as shown in Theorem 2. With this result we can see that an exact cutting plane algorithm gets in general very expensive as a large number of disjunctive hyperplanes has to be computed. Moreover we have not yet discussed how to determine a cut to the related  $k$ -disjunction. As intersection cuts according to basis relaxations do not generate strong cuts in general, this is an important issue for practical applications. On the other hand we have seen that a wide class of problems can even be solved using split cuts. Moreover the convergence properties of  $k$ -disjunctive cuts depend on the structure of the projection of the polyhedron and the objective function. This fact suggests to use information of the projection in cutting plane algorithms.

## 2.4 Computing $k$ -disjunctive cuts

At the end of this section we want to give an alternative to compute strong valid  $k$ -disjunctive cuts. Unlike the usual generating of valid cuts for MILP, we need as additional input the vector  $(c \ h)$  to which we want to cut. Moreover we restrict ourselves on shifted polyhedral cones  $P = \{(x, y) : Ax + Gy \leq b\}$  with apex  $(x^*, y^*)$ ,  $x^* \notin \mathbb{Z}^p$  and assume that the function  $cx + hy$  attains its unique maximum at  $(x^*, y^*)$  with value  $\gamma^*$ . In this situation we can describe  $(x^*, y^*)$  as polyhedron given by the (over-determined) system

$$P_{\gamma^*} := \left\{ \begin{pmatrix} A \\ -c \end{pmatrix} x + \begin{pmatrix} G \\ -h \end{pmatrix} y \leq \begin{pmatrix} b \\ -\gamma^* \end{pmatrix} \right\}. \quad (5)$$

Using Lemma 1, the projection of the above system on the  $x$ -space - that is equal to  $x^*$  - is given by

$$\text{proj}_X(P_{\gamma^*}) = \left\{ x \in \mathbb{R}^p : v^r \begin{pmatrix} A \\ -c \end{pmatrix} x \leq v^r \begin{pmatrix} b \\ -\gamma^* \end{pmatrix}, \forall r \in R \right\} \quad (6)$$

with  $R$  being the set of extreme rays of the cone

$$Q = \left\{ v \in \mathbb{R}^{m+1} : \begin{pmatrix} G \\ -h \end{pmatrix}^T v = 0, v \geq 0 \right\}. \quad (7)$$

As the cone  $Q$  is rational, we can assume that the extremal rays  $v^r$  are elements of the additive group

$$\mathcal{G}_{P_{\gamma^*}} := \left\{ w \in \mathbb{Q}^{m+1} : w \begin{pmatrix} A \\ -c \end{pmatrix} \in \mathbb{Z}^{m+1} \right\}. \quad (8)$$

Therewith we can use the above polyhedral description of  $\text{proj}_X(P_{\gamma^*})$  to define a valid  $k$ -disjunction for  $P$ , that does not contain the apex  $(x^*, y^*)$ . We do this by rounding up the right hand sides of the defining constraints of  $\text{proj}_X(P_{\gamma^*})$  in (6).

**Lemma 4.** *Let  $P, (x^*, y^*), (c \ h), \gamma^*$  and  $\text{proj}_X(P_{\gamma^*})$  as defined above. Moreover define for  $r \in R$*

$$\begin{aligned} d^r &:= v_{1,\dots,m}^r A - v_{m+1}^r c \\ \delta^r &:= \lfloor v_{1,\dots,m}^r b - v_{m+1}^r \gamma^* \rfloor + 1 \end{aligned}$$

with  $v^r = (v_{1,\dots,m}^r, v_{m+1}^r)$ . Then  $D(|R|, -d, -\delta)$  is a valid  $|R|$ -disjunction for  $P$  that does not contain  $(x^*, y^*)$ .

*Proof.* By definition it is  $\max\{cx + hy : (x, y) \in P_I\} < \gamma^*$ . So there is an  $\epsilon > 0$  such that  $cx + hy \leq \gamma^* - \epsilon$  is valid but not optimal for  $P_I$ . Moreover the inequalities defining  $\text{proj}_X(P_{\gamma^* - \epsilon})$  are given by  $v_{1,\dots,m}^r A - v_{m+1}^r c$  with right hand sides  $v_{1,\dots,m}^r b - v_{m+1}^r (\gamma^* - \epsilon)$ . The set  $\text{proj}_X(P_{\gamma^* - \epsilon})$  contains no integer points, so for all  $x \in \mathbb{Z}^p$  there is a  $r \in R$  with

$$d^r x > v_{1,\dots,m}^r b - v_{m+1}^r \gamma^*.$$

Therewith it follows that the polyhedron  $\{x \in \mathbb{R}^p : dx \leq \delta\}$  contains no integer point in its interior. This is equivalent to the set  $D(|R|, -d, -\delta)$  being a valid  $|R|$ -disjunction. Moreover, as the right hand side of each defining hyperplane of the projection has been enlarged by the definition of  $D$  it is obvious that  $(x^*, y^*)$  is not contained in the disjunction. This proves the lemma. We remark that the above definitions and the proof is similar to Lemma 2.  $\square$

So we have found a  $k$ -disjunction that can be used to cut off the current LP solution  $(x^*, y^*)$ . As we have mentioned at the beginning of this section we want to cut to the vector  $(c \ h)$ . We can do this now using the right hand side  $\delta$  of the disjunction  $D$ . As for  $v_{m+1}^r > 0$  the value of  $\delta^r$  depends on the objective function value, we can compute the objective function value that corresponds to the value of  $\delta^r$  that we have got by the rounding operation. The inequalities of  $D$  whose right hand sides  $\delta^r$  are independent of the value of  $\gamma$  can be omitted in this considerations. Taking the maximum of the related objective function values for all constraints gives us a valid cut to the vector  $(c \ h)$ . As the disjunction does not contain  $x^*$ , we can ensure that the current solution is cut off.

**Theorem 7.** *Let  $P, (x^*, y^*), (c \ h), \gamma^*, \text{proj}_X(P_{\gamma^*})$  and  $D(|R|, -d, -\delta)$  as defined above. Let for  $r \in R$  with  $v_{m+1}^r > 0$*

$$\gamma^r := \frac{\delta^r - v_{1,\dots,m}^r b}{-v_{m+1}^r}$$

and  $\hat{\gamma} = \max\{\gamma^r : r \in R, v_{m+1}^r > 0\}$ . Then  $cx + hy \leq \hat{\gamma}$  is valid for  $P_I$  and  $cx^* + hy^* > \hat{\gamma}$ .

*Proof.* The validity of the inequality  $cx + hy \leq \hat{\gamma}$  follows directly using Lemma 4, as it is a disjunctive cut according to  $D(|R|, -d, -\delta)$  by definition. Equally it follows that  $cx^* + hy^* > \hat{\gamma}$ .  $\square$

As we have finished the derivation of the  $k$ -disjunctive cut, we want to add some remarks. Using the projection as  $k$ -disjunction, we solve the problem how to find a suitable  $k$ -disjunction for cutting in general. This relates both to the selection of the number  $k$  and the selection of the defining hyperplanes of the disjunction. Moreover we have seen in the last subsections, that using information of the projection can be useful. On the other hand, the projection that we use corresponds to the predisposed cutting vector. So the selection of a suitable  $k$ -disjunction is partially shifted to the selection of the cutting vector. Here it is i.e. open how to choose cutting vectors to get deep cuts in general. However, for solving a given MILP we will see in the next section that this approach leads to a finite algorithm if we use the objective function vector.

### 3 Algorithm

We now turn to an algorithmic application of the previous results and want to present an exact algorithm that solves a bounded MILP in finite time. It is based on a series of mixed integer Gomory cuts that is mixed with certain  $k$ -disjunctive cuts which are required as discussed in subsection 2.3. The  $k$ -disjunctive cuts we use here are similar to the ones we introduced in subsection 2.4, using the objective function as the vector to which we cut. As the assumptions that we have made there for the  $k$ -disjunctive cuts are in general not satisfied, we have to do some modifications. So we will define  $k$ -disjunctive cuts over general polyhedra  $P$  for an arbitrary cut vector  $(c \ h)$ . We discuss the details of the generalization. It is clear that the equalities (6), (7), (8) also describe the projection for  $P$  being a general polyhedron and  $\gamma^*$  being an arbitrary value of the objective function. Even the derivation of a valid  $k$ -disjunction and a valid  $k$ -disjunctive cut, respectively, is true, if the value  $\gamma^*$  of the objective function  $cx + hy$  is not optimal for  $P_I$ . However, for the application in the algorithm we will define a slightly weaker version of the  $|R|$ -disjunctive cut that does not always cut off the current LP solution, but can be used more general. We do this in the next

**Theorem 8.** *Let  $P$  be a polyhedron and  $\gamma^*$  such that  $cx + hy \leq \gamma^*$  is valid for  $P_I$  but not for  $P$ . Define the  $|R|$ -disjunction  $D(|R|, -d, -\delta)$  using the equalities (5), (6), (7), (8) with*

$$\begin{aligned} d^r &:= v_{1,\dots,m}^r A - v_{m+1}^r c \\ \delta^r &:= \lceil v_{1,\dots,m}^r b - v_{m+1}^r \gamma^* \rceil \end{aligned}$$

and let  $\hat{\gamma} = \max\{\gamma^r : r \in R, v_{m+1}^r > 0\}$  analog to Theorem 7. Then  $cx + hy \leq \hat{\gamma}$  is a valid cutting plane for  $P_I$ .

*Proof.* By assumption  $\text{proj}_X(P_{\gamma^*})$  cannot contain an integral point in its interior. So rounding up the right hand sides gives a valid  $|R|$ -disjunction. Therefor  $cx + hy \leq \hat{\gamma}$  is a valid cutting plane analog to the proof of Theorem 7.  $\square$

We go on with the single steps of the algorithm. We start with the usual mixed integer Gomory algorithm as long as we get either a feasible solution of the MILP or a solution of the LP relaxation that has a lower objective function value. This happens in finite time by Corollary 1 if we restrict ourselves to polytopes. If the objective function value has decreased we can apply Theorem 8 and compute a valid  $|R|$ -disjunctive cut using the objective function as vector to which we cut. Now we can apply the Gomory algorithm to the modified program again, until either a feasible solution is found or the objective function value decreases, and use Theorem 8 again. In this way we get an algorithm that finitely terminates with an optimal solution to the given MILP or detects infeasibility. The formal algorithm is stated in Algorithm 1.

**Theorem 9.** *Let a bounded MILP (1) be given. Then Algorithm 1 either finds an optimal solution or detects infeasibility in a finite number of steps.*

*Proof.* The proof follows immediately with the following two facts: Every while loop (16) to (27) has only finite many iterations by Corollary 1 as  $P$  is bounded by presumption. Similarly the outer while loop (14) to (31) has only finite many iterations as the possible number of different values  $\hat{\gamma}$  is finite.  $\square$

Before we further discuss the algorithm we will give two examples. We start with repeating Example 1:

**Example 5.** *Let again the MILP*

$$\begin{aligned} \max y \\ -x_1 + y &\leq 0 \\ -x_2 + y &\leq 0 \\ x_1 + x_2 + y &\leq 2 \\ x_1, x_2 &\in \mathbb{Z} \end{aligned}$$

*with the optimal solution  $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$  of the LP relaxation be given. The mixed integer Gomory cuts according to  $x_1$  and  $x_2$  are given by  $-x_1 + 2y \leq 0$  and  $-x_2 + 2y \leq 0$  with the new LP solution  $(\frac{4}{5}, \frac{4}{5}, \frac{2}{5})$ . As the value of the objective function has decreased, we compute  $\tilde{\gamma}$  as in Theorem 8. The extremal rays of the cone  $\{(1 \ 1 \ 1 \ -1)y = 0, y \geq 0\}$  are the three vectors*

$$(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1),$$

*so the projection  $\text{proj}_X(P_{\tilde{\gamma}})$  is given by*

$$\begin{aligned} -x_1 &\leq 0 - \gamma \\ -x_2 &\leq 0 - \gamma \\ x_1 + x_2 &\leq 2 - \gamma \end{aligned}$$

---

**Algorithm 1** Exact cutting plane algorithm

---

```
1: Input: bounded MILP (1)
2: Output: "optimal solution  $(x^*, y^*)$ " or "problem infeasible" if no solution exists;
3:
4:  $(x^*, y^*) := \operatorname{argmax} \{cx + hy : (x, y) \in P\};$ 
5:  $\gamma^* := \max\{cx + hy : (x, y) \in P\};$ 
6:
7: if  $P = \emptyset$  then
8:   "problem infeasible"; break
9: end if
10: if  $x^* \in \mathbb{Z}^p$  then
11:   "optimal solution  $(x^*, y^*)$ "; break
12: end if
13:
14: while  $x^* \notin \mathbb{Z}^p$  do
15:    $\gamma := \gamma^*$ ;
16:   while  $\gamma^* = \gamma$  do
17:     Compute Gomory cut  $\alpha^1 x + \alpha^2 y \leq \beta$  to  $P, (x^*, y^*)$  by least index rule;
18:      $P := P \cap \{(x, y) : \alpha^1 x + \alpha^2 y \leq \beta\};$ 
19:      $(x^*, y^*) := \operatorname{argmax} \{cx + hy : (x, y) \in P\};$ 
20:      $\gamma^* := \max\{cx + hy : (x, y) \in P\};$ 
21:     if  $P = \emptyset$  then
22:       "problem infeasible"; break
23:     end if
24:     if  $x^* \in \mathbb{Z}^p$  then
25:       "optimal solution  $(x^*, y^*)$ "; break
26:     end if
27:   end while
28:
29:   Compute  $\hat{\gamma} = \max\{\gamma^r : r \in R, v_{m+1}^r > 0\}$  according to Theorem 8 for  $P, (c \ h), \gamma^*$ ;
30:    $\gamma^* = \hat{\gamma}$ ;
31: end while
32:
```

---

Inserting the current value  $\gamma = \frac{2}{5}$  of the objective function and rounding gives  $\max_{r=1,2,3} \tilde{\gamma}^r = \max\{0, 0, 0\} = 0$ . After applying the related cut  $y \leq 0$  we get as new LP solution the feasible point  $(2, 0, 0)$  and the algorithm stops with an optimal solution.

Second we show how the algorithm works for the example of Owen and Mehrotra [12]. For this ILP the usual mixed integer Gomory algorithm does not converge to the optimum.

**Example 6.** *Let the ILP*

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 8x_1 + 12x_2 \leq 27 \\ & 8x_1 + 3x_2 \leq 18 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

with the initial LP solution  $(\frac{15}{8}, 1)$  be given. After applying the first possible cut to  $x_1$  the value of the objective function decreases and we can go to the second step of the algorithm. As we have an ILP it is  $\text{proj}_X(P_\gamma) = P_\gamma$  with  $P_\gamma$  given by

$$\begin{aligned} -x_1 - x_2 & \leq -\gamma \\ 8x_1 + 12x_2 & \leq 27 \\ 8x_1 + 3x_2 & \leq 18 \\ x_1, x_2 & \geq 0 \end{aligned}$$

By rounding we get finally the valid cut  $x_1 + x_2 \leq 2$  that relates to the optimal objective function value.

The result of this example is typical for applying the algorithm on ILP. In this case we have to presume that all input data is integral and the  $k$ -disjunctive cut to the objective function reduces to the Chvátal Gomory cut of the objective function vector.

Concluding we want to discuss the algorithm. We have seen in the last example that for an ILP the  $k$ -disjunctive cut reduces to an integer Gomory cut to the objective function. So the whole algorithm can be seen as a variant of the pure integer Gomory algorithm in this case. The crucial fact for finite convergence of the integer algorithm is the possibility to add cuts both to the objective function and to each variable if they are not integral. Using  $k$ -disjunctive cuts to the objective function we have now the possibility to add cuts to the objective function in the case of MILP as well. Therewith we obtain a convergent algorithm in analogy to the integer case.

Of course the complex part of the algorithm consists in computing the  $k$ -disjunctive cut as the number  $|R|$  of extreme rays  $v^r$  of the cone  $Q$  grows exponentially. So an efficient algorithm for computing the extreme rays of the related cone is required. Moreover



we have to ensure that the computed rays satisfy the integrality constraints, i.e. are contained in the group  $\mathcal{G}_{P_{\gamma^*}}$ . Therefor we can presume in practical applications the coefficients of the matrix  $A$  and the vector  $c$  to be integer. Then the integrality constraints are satisfied, if all of the extreme rays  $v^r$  are integer. However, we will not further deal with this issue here, but refer to the papers of Henk and Weismantel [9] and of Hemmecke [8] and the references therein. They state several algorithms for this and the similar problem of computing Hilbert bases of polyhedral cones.

At last we want to give a further interpretation of the algorithm. Therefor we assume that the feasible domain  $P$  is full dimensional and bounded. One can see that in this case we can always choose an optimal solution of the MILP such that  $q$  defining inequalities of  $P$  are active. So the solution is contained in a  $(p + q) - q = p$ -dimensional face of  $P$ . Therefor we can solve the MILP by solving each of the related  $p$ -dimensional subproblems and taking the best solution. Moreover the set of feasible solutions in each  $p$ -dimensional face is discrete in general, so solving a MILP for a  $p$ -dimensional face can be interpreted as solving an ILP, as we could apply a suitable affine transformation. This means that solving a MILP can be seen as parallel solving of several ILP. Especially every valid cutting plane for  $P_I$  is even valid for each of the discrete subproblems. Therefor we need information of the related discrete subproblems if we want to generate strong valid cuts. As the number of  $p$ -dimensional faces of  $P$  grows exponentially, this interpretation also gives another reasoning that we need  $k$ -disjunctive cuts with an exponential number of defining disjunctive hyperplanes to solve general MILP. Within the algorithm we can find the  $p$ -dimensional subproblems in the facets of the polyhedron  $\text{proj}_X(P_{\gamma^*})$ , where the value of the right hand side  $\delta^i$  can be related to the current objective function value of the subproblem.

## References

- [1] K. Andersen, G. Cornuéjols, and Y. Li. Split closure and intersection cuts. *Mathematical Programming*, A 105:457–493, 2005.
- [2] E. Balas. Intersection cuts - a new type of cutting planes for integer programming. *Operations Reserach*, 19:19–39, 1971.
- [3] E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58:295 – 324, 1993.
- [4] E. Balas and A. Saxena. Optimizing over the split closure. *Mathematical Programming*, A:to appear.
- [5] W. Cook, R. Kannan, and A. Schrijver. Chvátal closures for mixed integer programming. *Mathematical Programming*, A 93:155–174, 1990.
- [6] G. Cornuéjols and Y. Li. Elementary closures for integer programs. *Operations Research Letters*, 28:1–8, 2001.

- [7] R. E. Gomory. An algorithm for integer solutions to linear programs. In R.L. Graves and P. Wolfe, editors, *Recent Advances in Mathematical Programming*, pages 269–303. McGraw-Hill, New York, 1963.
- [8] Raymond Hemmecke. On the computation of hilbert bases and extreme rays of cones. *www.arXiv.org*, *arXiv:math.CO/0203105 v1*, 2006.
- [9] M. Henk and R. Weismantel. On hilbert bases of polyhedral cones. In *ZIB Collection*, number SC96-12. April 1996.
- [10] G.L. Nemhauser and L.A. Wolsey. *Integer and Combinatorial Optimization*. Wiley, New York, 1988.
- [11] G.L. Nemhauser and L.A. Wolsey. A recursive procedure to generate all cuts for 0-1 mixed integer programs. *Mathematical Programming*, 49:379–390, 1990.
- [12] J.H. Owen and S. Mehrotra. A disjunctive cutting plane procedure for general mixed-integer linear programs. *Mathematical Programming*, A 89:437 – 448, 2001.